Fast, Simple and Accurate Computation of the Currents on an Arbitrarily Large Circular Loop Antenna

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Abstract—**This paper is presenting a very efficient method for computing the currents on a circular loop antenna with arbitrarily large size. Pocklington's integral equation is formulated for the circular loop geometry, and the method of moments with point matching is applied to cast the equation's discrete counterpart. The basis functions are chosen in such a way that the relevant square matrix is circulant, and therefore amenable to exact eigenvalue analysis. Subsequently, the matrix is diagonalized and inverted analytically, yielding simple, analytical expressions for the weights of the basis functions, and hence the current and the input admittance of the antenna. The algorithm utilizes mainly elementary mathematical functions, as opposed to already known expressions in the literature, and yields results in the form of a rapidly convergent, single series, applicable to extremely large loops.**

Index Terms—**Circulant matrices, current, input admittance, loop antennas, method of moments (MoM).**

I. INTRODUCTION

THE circular loop antenna has been one of the most popular radiators to analyze, with mathematical expressions and computational results presented in several pioneering papers [1]–[5]. It is long well-known that for very small loops the currents is almost constant, greatly simplifying the derivation of the radiated field expressions [2]. As the loop radius increases, the current distribution becomes initially sinusoidal [6], and finally incorporates a large number of Fourier modes. The estimation of the current distribution and the input impedance/admittance of an arbitrarily sized loop has been the main focus of several works in the literature. The only way to derive any analytical expressions so far was the Fourier expansion of the current, which is clearly periodic. Invoking complex exponentials as basis functions in a method of moments (MoM) formulation [7] yields a diagonal system matrix, which is trivial to invert. Early papers on the topic faced convergence problems for the series derived. In [1], Hallen concluded that the series was divergent, due to a singularity occurring after the inclusion of too many terms. A few years later, Storer [3] avoided the large term contribution by replacing the series by an integral, which he evaluated via complex analysis techniques. However, the reliability of the results was questioned in [4], where alternative approximations were proposed to circumvent this disagreeable situation. Analytical expressions were derived in [4] for the weights of the Fourier terms, which, however, are rather formidable, given only symbolically in terms of integrals of Lommel-Weber, Bessel and Modified Bessel functions. These expressions are considered classical, and have also been adopted by standard textbooks, such as [7]. Although exact, they are generally difficult to compute, and are applicable in practice only for small to medium-sized loops. In the open literature, the largest loops amenable to this approach extend to six wavelengths in circumference [8]. These elaborate expressions are still unavoidable in [9], where a magnetic frill current model is used for the excitation.

Alternatively, the MoM [7] can be invoked with standard basis and testing functions, resulting in a full square matrix, as opposed to the diagonal matrix required by the Fourier expansion. The algorithm is completely numerical, involving no special functions, however for very large loops the number of unknowns rises dramatically, rendering the method cumbersome and inefficient. Fast methods [10] can obviously be employed to accelerate the MoM algorithm, however the unknown current is still computed through a numerical matrix inversion, since no closed form expressions can be derived. Therefore, ill-conditioning may deplete the methods' efficiency for very large loops, since the number of iterations required for the linear system solution may increase significantly. Convergence checks are thus laborious, whereas loop design becomes a very tedious task, since several time-consuming repetitions of the algorithm may be required to obtain the desirable antenna parameters.

All these complications could be resolved if simple, analytic or semi-analytic expressions could somehow be derived for the current and the input impedance of the antenna. In this paper it is shown that such expressions can be extracted indeed, by invoking the properties of circulant matrices [11]. This type of matrices can be diagonalized and inverted in closed form, irrespective of their dimension. The important implications of this property can be readily seen in scattering analysis of structures with circular symmetry [12]–[14]. In particular, [12] elaborates on the use of block-circulant matrices for the radiation analysis of circular arrays of dipoles, [13] utilizes similar concepts for the scattering analysis of discrete bodies of revolution, whereas [14] is a representative of a series of papers by the same research group on the optimization of the Method of Auxiliary Sources applied to circular cylinders of various materials. The geometry of the circular loop is clearly amenable to such an approach, provided that the basis and testing functions of the MoM formulation are suitably chosen. It will be proven that both the current

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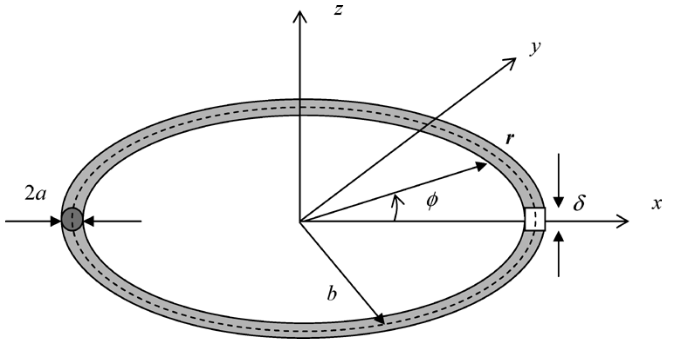

Fig. 1. Geometry of a thin circular loop antenna.

and the input admittance of the loop can be given in terms of a rapidly convergent, single series, which involves almost exclusively elementary functions. The advantages of such expressions cannot be overemphasized, since all aforementioned drawbacks of other methods are completely circumvented. Accurate and robust calculation of the antenna parameters for loop radii of hundreds of wavelengths becomes feasible in a few seconds, even on a scientific pocket calculator. Furthermore, convergence checks are trivial to perform, leading to high precision of the numerical values obtained.

The structure of the paper is as follows: Section II formulates the MoM system in a suitable way, the matrix is inverted exactly, and semi-analytical expressions for the current and the input admittance are derived. Section III presents several numerical results for loop circumference up to 100 wavelenths, while comparisons with reference data for small loops validate the algorithm and the computational code. Section IV outlines and concludes the paper.

A $e^{j\omega t}$ time dependence is assumed and suppressed throughout the paper.

II. Analytical Formulation

A. Matrix Fill

Assume a thin circular loop of radius b and wire radius $a \ll b$, driven, without loss of generality, by a delta gap generator (Fig. 1). Let the unknown current on the loop be $I(\phi)$, which is assumed to flow on the wire axis of radius b, given that the wire is very thin. The azimuthal component $E^r_\varphi(\boldsymbol{r})$ of the radiated electric field at an arbitrary observation point $\boldsymbol{r}(r,\phi)$ is given by [7]

$$E^r_\phi(\boldsymbol{r}) = -j\omega\mu r' \int_0^{2\pi} I(\phi')\cos(\phi-\phi')\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}d\phi' + \frac{1}{j\omega\varepsilon r}\int_0^{2\pi}\frac{dI(\phi')}{d\phi'}\frac{d}{d\phi}\left[\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\right]d\phi' \tag{1}$$

where, as usual, $|\boldsymbol{r}-\boldsymbol{r}'|$ is the distance between source and observation points, ω is the circular frequency, ε is the dielectric permittivity of the medium, μ is the magnetic permeability and k is the wavenumber. Applying the appropriate boundary condition for the electric field at a discrete set of N points (point matching) with azimuthal positions $\phi_m = (2\pi/N)m \equiv m\widetilde{\phi}$, $m = 1,\ldots,N$ on the outer surface of the wire yields

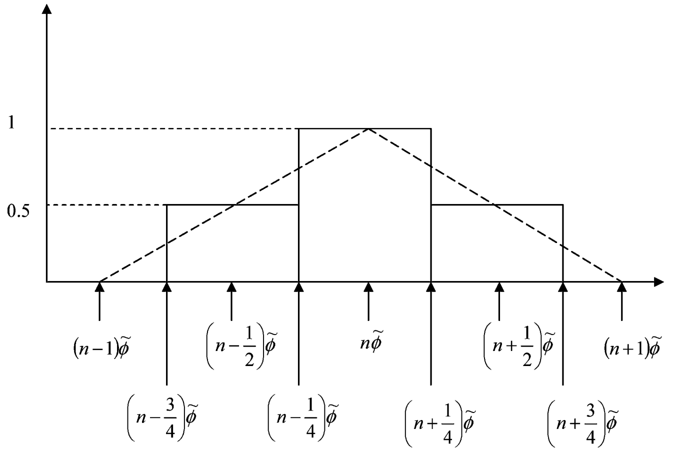
1
0.5
$(n-1)\widetilde{\phi}$
$\left(n-\frac{1}{2}\right)\widetilde{\phi}$
$n\widetilde{\phi}$
$\left(n+\frac{1}{2}\right)\widetilde{\phi}$
$(n+1)\widetilde{\phi}$
$\left(n-\frac{3}{4}\right)\widetilde{\phi}$
$\left(n-\frac{1}{4}\right)\widetilde{\phi}$
$\left(n+\frac{1}{4}\right)\widetilde{\phi}$
$\left(n+\frac{3}{4}\right)\widetilde{\phi}$

Fig. 2. The step-pulse basis function, which adequately approximates a triangular basis function (dashed line).

$$(b+a)E^i_\phi(\phi_m) = jk\eta b(b+a)\int_0^{2\pi} I(\phi')\cos(\phi_m-\phi')\frac{e^{-jkR}}{4\pi R}d\phi' - \frac{\eta}{jk}\frac{\partial}{\partial\phi}\int_0^{2\pi}\frac{dI(\phi')}{d\phi'}\frac{e^{-jkR}}{4\pi R}d\phi'\Bigg|_{\phi=\phi_m} \tag{2}$$

where $E^i_\phi(\phi_m)$ is the azimuthal component of the incident electric field due to the antenna excitation calculated at ϕ_m

$$R \equiv b\sqrt{4\sin^2\left(\frac{\phi-\phi'}{2}\right)+\left(\frac{a}{b}\right)^2} \tag{3}$$

and η is the medium intrinsic impedance.

To complete the MoM formulation, and cast the linear system of equations, the current $I(\phi)$ must be expanded into a weighted sum of basis functions, i.e.

$$I(\phi') \cong \sum_{n=1}^{N} I_n F_n(\phi'). \tag{4}$$

It should be kept in mind that the purpose in our procedure is to obtain a circulant matrix, therefore the basis functions $F_n(\phi')$ should be chosen accordingly. Pulse basis functions would be suitable from this aspect, but the accuracy of the final result would be rather poor [7]. Triangular basis functions would guarantee higher accuracy, however they would unnecessarily complicate the matrix elements calculations. A very good compromise between accuracy and simplicity is the step-pulse basis functions [7], which adequately approximate a triangular basis function, yet maintaining the salient features of a regular pulse (see Fig. 2). The analytical expression for the step pulse of Fig. 2 is

$$F_n(\phi') \equiv \Pi_n(\phi') + \frac{1}{2}\Pi_{n-\frac{1}{2}}(\phi') + \frac{1}{2}\Pi_{n+\frac{1}{2}}(\phi') \tag{5}$$

where

$$\Pi_n(\phi') \equiv \begin{cases} 1, & \left(n-\frac{1}{4}\right)\widetilde{\phi} \le \phi' \le \left(n+\frac{1}{4}\right)\widetilde{\phi} \\ 0, & \text{else.} \end{cases} \tag{6}$$

From (2) and (4) it becomes evident that the derivatives of $F_n(\phi')$ must also be utilized in the formulation of the linear system. Using the properties of generalized functions (distributions), (5) and (6) yield

$$\begin{aligned}\frac{dF_n(\phi')}{d\phi'} =& \frac{1}{2}\delta\left(\phi' - \left(n-\frac{3}{4}\right)\widetilde{\phi}\right) + \frac{1}{2}\delta\left(\phi' - \left(n-\frac{1}{4}\right)\widetilde{\phi}\right)\\ &-\frac{1}{2}\delta\left(\phi' - \left(n+\frac{1}{4}\right)\widetilde{\phi}\right) - \frac{1}{2}\delta\left(\phi' - \left(n+\frac{3}{4}\right)\widetilde{\phi}\right)\end{aligned} \tag{7}$$

where $\delta(\bullet)$ is the Dirac's delta function. Again, using the properties of the delta function, and approximating the $\partial/\partial\phi$ derivatives by finite differences, the second integral in (2) is approximately equal, due to (4), to a weighted sum of the following expressions:

$$\begin{aligned}&\frac{\partial}{\partial\phi}\int_0^{2\pi}\frac{dF_n(\phi')}{d\phi'}\frac{e^{-jkR}}{4\pi R}d\phi'\Bigg|_{\phi=\phi_m}\\ &\cong \frac{1}{8\pi\widetilde{\phi}}\left[\frac{\exp\left\{-jkR_{m+\frac{1}{2},n-\frac{3}{4}}\right\}}{R_{m+\frac{1}{2},n-\frac{3}{4}}} - \frac{\exp\left\{-jkR_{m-\frac{1}{2},n-\frac{3}{4}}\right\}}{R_{m-\frac{1}{2},n-\frac{3}{4}}}\right]\\ &+\frac{1}{8\pi\widetilde{\phi}}\left[\frac{\exp\left\{-jkR_{m+\frac{1}{2},n-\frac{1}{4}}\right\}}{R_{m+\frac{1}{2},n-\frac{1}{4}}} - \frac{\exp\left\{-jkR_{m-\frac{1}{2},n-\frac{1}{4}}\right\}}{R_{m-\frac{1}{2},n-\frac{1}{4}}}\right]\\ &-\frac{1}{8\pi\widetilde{\phi}}\left[\frac{\exp\left\{-jkR_{m+\frac{1}{2},n+\frac{1}{4}}\right\}}{R_{m+\frac{1}{2},n+\frac{1}{4}}} - \frac{\exp\left\{-jkR_{m-\frac{1}{2},n+\frac{1}{4}}\right\}}{R_{m-\frac{1}{2},n+\frac{1}{4}}}\right]\\ &-\frac{1}{8\pi\widetilde{\phi}}\left[\frac{\exp\left\{-jkR_{m+\frac{1}{2},n+\frac{3}{4}}\right\}}{R_{m+\frac{1}{2},n+\frac{3}{4}}} - \frac{\exp\left\{-jkR_{m-\frac{1}{2},n+\frac{3}{4}}\right\}}{R_{m-\frac{1}{2},n+\frac{3}{4}}}\right]\end{aligned} \tag{8}$$

where

$$R_{\kappa,\nu} \equiv b\sqrt{4\sin^2\frac{(\kappa-\nu)\widetilde{\phi}}{2} + \left(\frac{a}{b}\right)^2} \tag{9}$$

From (8) and (9) it is clear that this portion of the MoM matrix elements, corresponding to the second integral in (2) is a N-periodic function of the difference $m-n$ only, and therefore demonstrates circulant behavior, as required. Similarly, the first integral in (2) can be approximately calculated. In the case where $m \neq n$, the evaluation is straightforward, yielding

$$\begin{aligned}&\int_0^{2\pi}F_n(\phi')\cos(\phi_m-\phi')\frac{e^{-jkR}}{4\pi R}d\phi'\\ &\cong \frac{\widetilde{\phi}}{8\pi}\frac{\exp\{-jkR_{m,n}\}}{R_{m,n}}\cos(m-n)\widetilde{\phi}\\ &+\frac{\widetilde{\phi}}{16\pi}\frac{\exp\left\{-jkR_{m,n-\frac{1}{2}}\right\}}{R_{m,n-\frac{1}{2}}}\cos\left(m-n+\frac{1}{2}\right)\widetilde{\phi}\\ &+\frac{\widetilde{\phi}}{16\pi}\frac{\exp\left\{-jkR_{m,n+\frac{1}{2}}\right\}}{R_{m,n+\frac{1}{2}}}\cos\left(m-n-\frac{1}{2}\right)\widetilde{\phi}\\ &\quad m \neq n\end{aligned} \tag{10}$$

which is again a function of $m-n$ only. For $m=n$ a much more refined approach is required, due to the behavior of the Green's function kernel, and the resulting very small values of the denominator. The two last terms in the right-hand side of (10), which correspond to the side pulses in Fig. 2 are still valid, however the first term, which corresponds to the central pulse, must be evaluated with higher precision. Local linearization of the wire, as proposed in [7], turns out to yield inaccurate results. On the contrary, a more careful integration in polar coordinates yields ([15, p. 173, 2.597,1]

$$\begin{aligned}&\int_0^{2\pi}\Pi_m(\phi')\cos(\phi_m-\phi')\frac{e^{-jkR}}{4\pi R}d\phi'\\ &\cong \frac{1}{4\pi}\int_{\left(m-\frac{1}{4}\right)\widetilde{\phi}}^{\left(m+\frac{1}{4}\right)\widetilde{\phi}}\left(\frac{1}{R}-jk\right)d\phi'\\ &= \frac{1}{\pi a\sqrt{1+\tau^2}}F\left(\sigma, \frac{\tau}{\sqrt{1+\tau^2}}\right) - \frac{jk\widetilde{\phi}}{8\pi}\end{aligned} \tag{11}$$

where R is expressed as in (3), evaluated at $\phi = \phi_m$, and

$$\sigma \equiv \arcsin\frac{\sqrt{1+\tau^2}\sin\frac{\widetilde{\phi}}{8}}{\sqrt{1+\tau^2\sin^2\frac{\widetilde{\phi}}{8}}}, \quad \tau \equiv \frac{2b}{a}$$

$$F(\varphi,\kappa) \equiv \int_0^{\varphi}\frac{da}{\sqrt{1-\kappa^2\sin^2 a}} \quad \text{(elliptic integral of first kind).} \tag{12}$$

The elliptic integral encountered in (12) is the only nonelementary function involved in the entire formulation, however it is very easily computable through efficient and simple subroutines, readily available in the literature, e.g. [16, p. 260].

In brief, the MoM linear system produced by (2) is written on the basis of (8)–(12) as

$$\{\boldsymbol{V}\} = [\boldsymbol{Z}]\{\boldsymbol{I}\} \tag{13}$$

where $\{\boldsymbol{I}\}$ is the vector of unknown weights, $V_n \equiv (b+a)E_\phi^i(n\widetilde{\phi})$ and $[\boldsymbol{Z}]$ is a square matrix, whose elements are N-periodic functions of the index difference $p \equiv m-n$ only. This is the defining property of a circulant matrix, namely $[\boldsymbol{Z}] = circ(z_1, z_2, \ldots, z_N)$ with

$$\begin{aligned}z_{p+1} =& jk\eta b(b+a)\\ &\times\left[\Psi_p + \frac{\widetilde{\phi}}{4}\frac{\exp\left\{-jkR_{p+\frac{1}{2}}\right\}}{4\pi R_{p+\frac{1}{2}}}\cos\left(p+\frac{1}{2}\right)\widetilde{\phi}\right.\\ &\left.+\frac{\widetilde{\phi}}{4}\frac{\exp\left\{-jkR_{p-\frac{1}{2}}\right\}}{4\pi R_{p-\frac{1}{2}}}\cos\left(p-\frac{1}{2}\right)\widetilde{\phi}\right]\\ &+j\frac{\eta}{k}\frac{1}{2\widetilde{\phi}}\left[\frac{\exp\left\{-jkR_{p+\frac{5}{4}}\right\}}{4\pi R_{p+\frac{5}{4}}} + \frac{\exp\left\{-jkR_{p+\frac{3}{4}}\right\}}{4\pi R_{p+\frac{3}{4}}}\right.\\ &\left.-2\frac{\exp\left\{-jkR_{p+\frac{1}{4}}\right\}}{4\pi R_{p+\frac{1}{4}}}\right]\end{aligned}$$

$$+j\frac{\eta}{k}\frac{1}{2\widetilde{\phi}}\left[\frac{\exp\left\{-jkR_{p-\frac{5}{4}}\right\}}{4\pi R_{p-\frac{5}{4}}}+\frac{\exp\left\{-jkR_{p-\frac{3}{4}}\right\}}{4\pi R_{p-\frac{3}{4}}}\right.$$
$$\left.-2\frac{\exp\left\{-jkR_{p-\frac{1}{4}}\right\}}{4\pi R_{p-\frac{1}{4}}}\right]$$
$$p=0,\ldots,N-1 \tag{14}$$

where

$$\Psi_p=\begin{cases}\frac{\widetilde{\phi}}{2}\frac{\exp\{-jkR_p\}}{4\pi R_p}\cos p\widetilde{\phi} & p\neq 0\\ \frac{1}{\pi a\sqrt{1+\tau^2}}F\left(\sigma,\frac{\tau}{\sqrt{1+\tau^2}}\right)-\frac{jk\widetilde{\phi}}{8\pi} & p=0\end{cases} \tag{15}$$

and

$$R_\kappa\equiv b\sqrt{4\sin^2\frac{\kappa\widetilde{\phi}}{2}+\left(\frac{a}{b}\right)^2}. \tag{16}$$

B. Analytical Solution of the Linear System

Since $[\boldsymbol{Z}]$ is circulant, its eigenvalues λ_q and eigenvectors $\{\boldsymbol{g}\}_q$ are given by the theorem in Appendix A, therefore $[\boldsymbol{Z}]$ can be explicitly diagonalized, and therefore analytically inverted. Elementary algebraic manipulation, similar to [14], based on the expressions of Appendix A, yields the exact solution of (13)

$$I_m=\frac{1}{N}\sum_{n=1}^{N}\sum_{q=1}^{N}\frac{1}{\lambda_q}\exp\left\{j(m-n)(q-1)\frac{2\pi}{N}\right\}V_n,$$
$$m=1,\ldots,N \tag{17}$$

for arbitrary excitation $\{\boldsymbol{V}\}$. For a delta source of voltage V across the gap, located at $\phi=0$, evidently

$$V_n=(b+a)E^i_\phi(n\widetilde{\phi})\delta_{n,1}=(b+a)\frac{V}{\frac{2\pi b}{N}}\delta_{n,1}=\left(1+\frac{a}{b}\right)\frac{VN}{2\pi}\delta_{n,1} \tag{18}$$

where $\delta_{n,1}$ is the Kronecker delta, hence (17) is simplified as

$$I_m=\frac{V}{2\pi}\left(1+\frac{a}{b}\right)\sum_{q=1}^{N}\frac{1}{\lambda_q}\exp\left\{j(m-1)(q-1)\frac{2\pi}{N}\right\} \tag{19}$$

and the input admittance is therefore given by

$$Y_{in}\equiv\frac{I_1}{V}=\frac{1}{2\pi}\left(1+\frac{a}{b}\right)\sum_{q=1}^{N}\frac{1}{\lambda_q} \tag{20}$$

The physical meaning of (20) is that the loop antenna can be characterized as a circuit of parallel impedances, equal to the eigenvalues of the MoM system (13) within a constant. Expressions (17), (19), and (20) constitute a great simplification with respect to standard formulas for the current and the input admittance given in [3], [4], and [7], since almost all calculations can be performed through single sums of elementary functions. It is pointed out again, as in the introduction, that the respective expressions in [3], [4], and [7] utilize complicated integrals of special functions that are cumbersome to compute, especially for large loops. Admittedly, the method proposed herein is not purely analytical, in a sense that the resulting expressions are neither "exact," nor "elegant," such as the ones in [3], [4], and [7]. However it will be shown in the next section that, from a computational point of view, (19) and (20) are much more efficient and highly accurate.

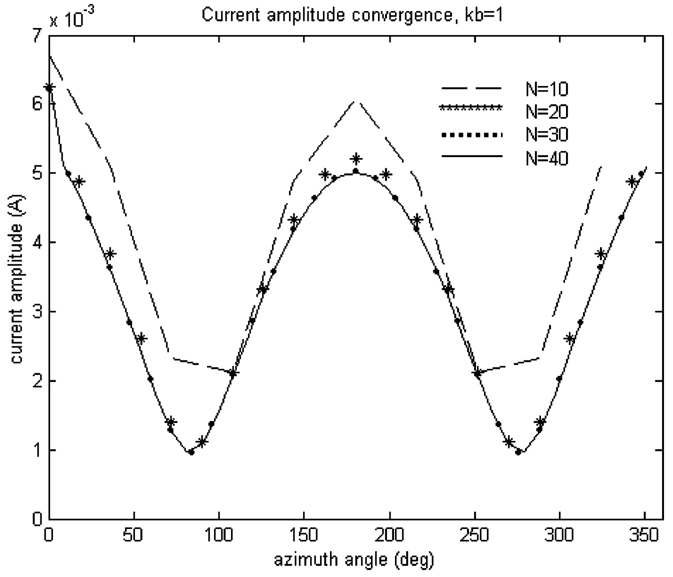

Fig. 3. Convergence of the current amplitude for a loop with $kb=1$ and $b/a=15$.

Furthermore, the capability of analytical matrix inversion, and the consequent insensitivity to matrix conditioning, offers to the method several unique advantages over purely numerical algorithms. Indeed, unlike the formulation proposed in this paper, ill-conditioning may decrease the accuracy of numerical methods combined with direct solvers, and reduce the convergence rates of methods based on iterative solvers. In terms of memory requirements, the demands are obviously very low $(O(N))$. Finally, in terms of computational cost, the entire procedure is at least comparable to fast methods, even in the ideal case for the latter, i.e. when a single iteration is sufficient. This is evident from (A1), (17), and (19), which essentially represent discrete Fourier transforms, and which can rapidly be calculated through FFTs, thus reducing the complexity of the algorithm from brute-force $O(N^2)$ down to $O(Nlog_2N)$. Outperforming fast methods, however, the result is given in only one step.

III. NUMERICAL RESULTS

To validate the analysis presented in Section II, several numerical results are extracted and compared to reference data, whenever available. First of all, the convergence rate of the series expressions involved is examined. Fig. 3 depicts the convergence behavior for the current amplitude on a loop with $kb=1$ (i.e. with circumference equal to λ) and $b/a=15$. The convergence behavior for the current phase is shown in Fig. 4. In both figures the horizontal axis corresponds to the azimuth angle along the loop circumference. From both figures it is concluded that convergence is achieved for a number of unknowns approximately equal to $N=40$, i.e. for an element arc length equal to $\lambda/40$. Of course, the radiation patterns are not as sensitive to the

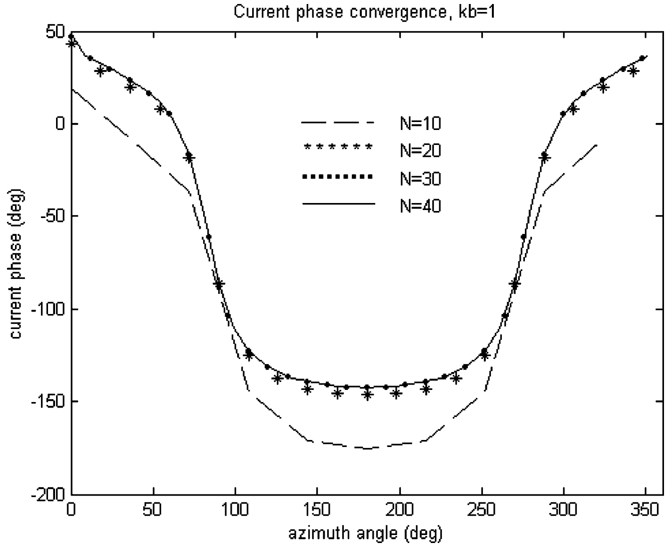

Fig. 4. Convergence of the current phase for a loop with $kb = 1$ and $b/a = 15$.

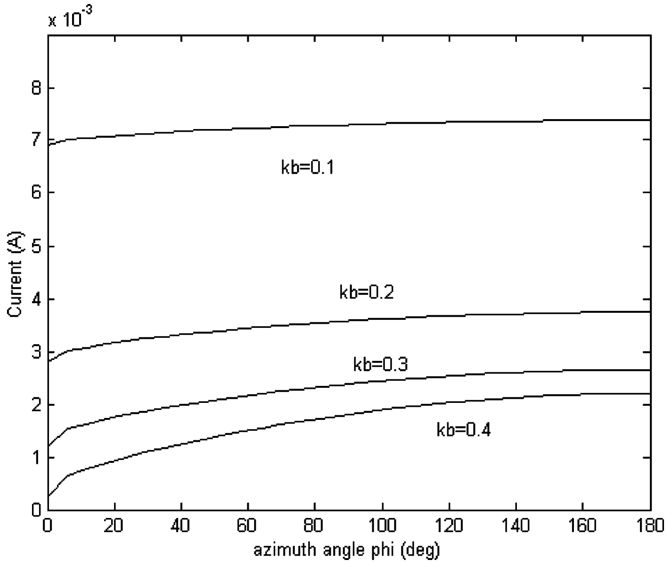

Fig. 5. Current amplitude for loops with various circumferences and $b/a = 23.62$. Compare with [18, p. 185, Fig. 5.7].

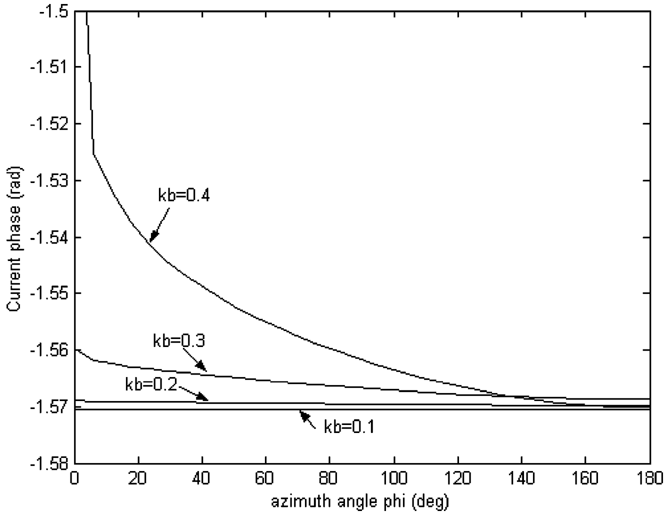

Fig. 6. Current phase for loops with various circumferences and $b/a = 23.62$. Compare with [18, p. 185, Fig. 5.7].

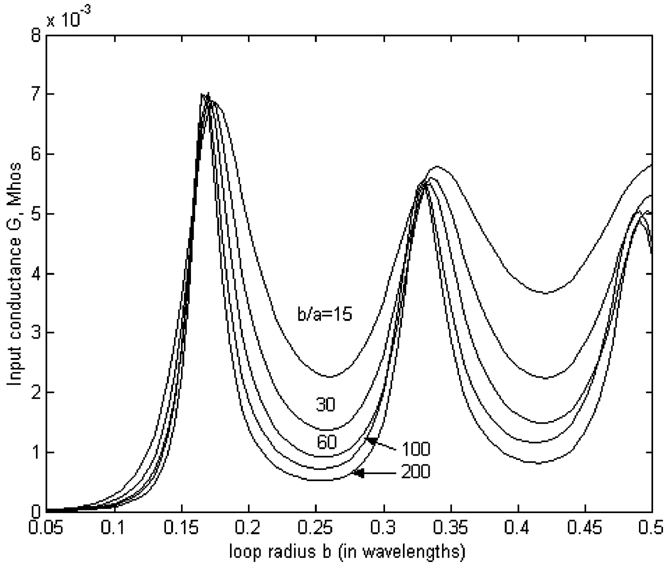

Fig. 7. Input conductance as a function of the loop radius, for various ratios b/a. Compare with [7, p. 93, Fig. 5.5].

element arc length as the actual currents [17], and they usually converge must faster, typically for element arc lengths equal to $\lambda/10$.

Comparison with reference results is performed by applying the analysis to loops of increasing circumference (ranging from 0.1 to 0.4 wavelengths) and radii ratio equal to $b/a = 23.62$ (or $2\ln(2\pi b/a) = 10$, to comply with the notation used in [3] and [18]). The results for the amplitude are shown in Fig. 5 and for the phase in Fig. 6, and are clearly identical to [18, p. 185, Fig. 5.7], borrowed in turn from [3]. Moreover, results for the input admittance are depicted in Figs. 7 (conductance) and 8 (susceptance). Fig. 7 is identical to [7, p. 93, Fig. 5.5], which was plotted using the standard, cumbersome expressions. Fig. 8 is very similar to [7, p. 93, Fig. 5.6], although not identical. The rather insignificant discrepancy is fully explained by the fact that the input susceptance for a delta gap model cannot be rigorously defined, since it approaches infinity as the gap width vanishes [5]. Therefore, any susceptance calculations strictly do not converge [7], and are meaningful only as approximate *estimates* for realistic loops [5] with excitations modeled by small, but non-vanishing, gaps.

Since all comparisons with reference data showed excellent agreement, validating the analysis, the semi-analytical results in Section II will now be used for current computations of very large loops. Figs. 9 and 10, depict the current amplitude and phase respectively, as a function of the azimuth angle along the loop circumference, for $a = \lambda/100$ and kb =10, 25, 50, and 100. Convergence was carefully examined, being achieved for N = 600, 2500, 5000, and 8000, respectively. Finally, Fig. 11 depicts the dependence of the input conductance as a function of the loop circumference, the latter extending to very large loops. Input susceptance is not plotted for reasons explained earlier. All calculations, including convergence checks, were performed on a typical PC in a matter of a few seconds, based on an elementary computer code. Hence, the method is valuable for very

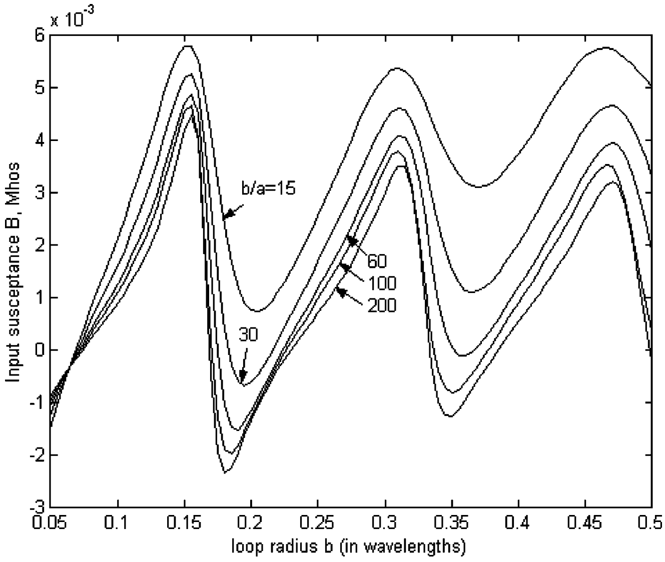

Fig. 8. Input susceptance as a function of the loop radius, for various ratios b/a. Compare with [7, p. 93, Fig. 5.6].

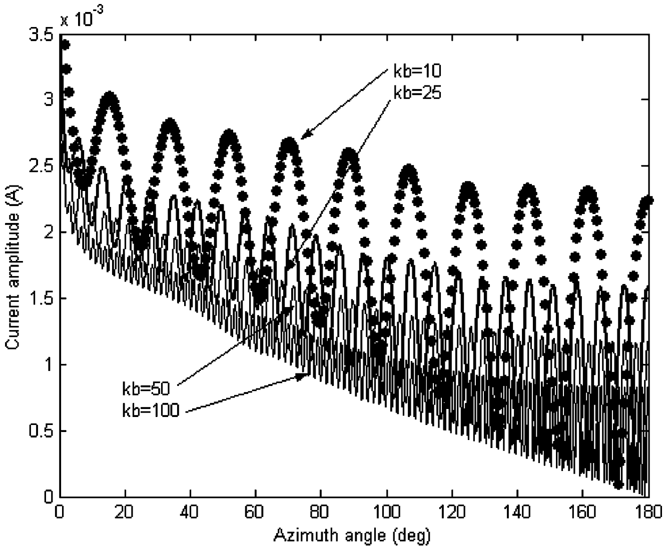

Fig. 9. Current amplitude as a function of the azimuth angle for loops with various, very large circumferences and $a = \lambda/100$.

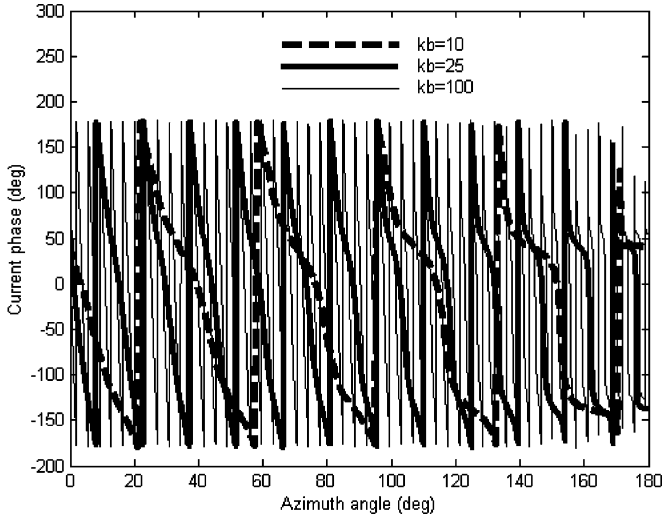

Fig. 10. Current phase as a function of the azimuth angle for loops with various, very large circumferences and $a = \lambda/100$.

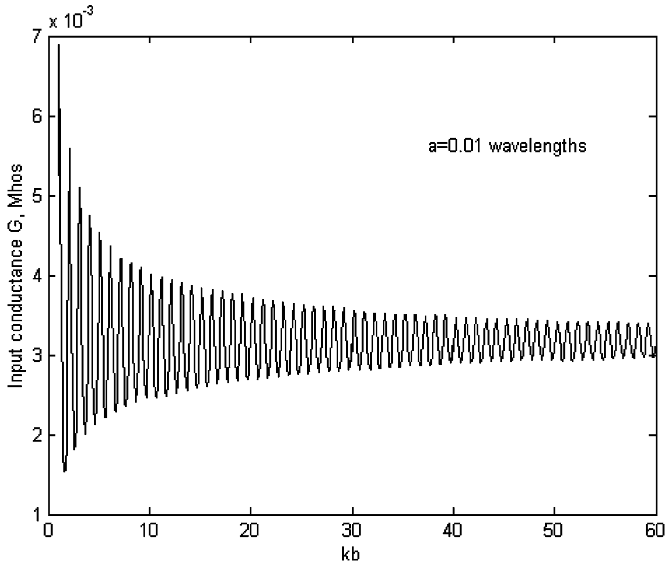

Fig. 11. Input conductance as a function of the loop circumference, extending to very large loops and $a = \lambda/100$.

large loop designs, since it is very simple to formulate, and since it retains the accuracy of the purely numerical MoM, requiring, though, very low memory and CPU time resources, and also avoiding instability problems caused by possible matrix ill-conditioning. The results shown in Figs. 9–11 are related to loop sizes far larger than the 6λ previous upper bound, recently documented in the literature [8].

IV. Summary and Conclusions

A semi-analytical formulation for the efficient and accurate calculation of the currents on extremely large loops was presented. The basis of the analysis was the MoM, whose linear system was appropriately cast in such a way that the relevant matrix was of circulant form. The matrix was therefore diagonalized and inverted by an exact procedure, yielding analytical expressions for the currents and the input admittance. The formulation utilized only elementary mathematical concepts, as opposed to standard analytical methods, previously developed in the literature, which involve complicated integrals of special functions. Moreover, the analytical invertibility of the MoM matrix reduces tremendously the MoM computational cost, with respect to memory and CPU time requirements, a very useful feature for the analysis and design of very large loops, where repetitive convergence checks are computationally very expensive and tedious. Furthermore, as opposed to fast methods, the final results cannot possibly be affected by ill-conditioning of the MoM matrix and hence, are always numerically stable. The method is not restricted by any upper bound of the size of the loop, and accurate data for circumferences up to 100 wavelengths were plotted, i.e. for loop sizes much larger than the 6 wavelength limit documented in the literature so far.

APPENDIX A

Theorem [11]: Let $[\boldsymbol{A}]$ be a $N \times N$ circulant complex matrix, i.e. $[\boldsymbol{A}] = circ(a_1, a_2, \ldots, a_N)$. Then $[\boldsymbol{A}]$ is normal, and thus diagonalizable. Its eigenvalues are given by

$$\lambda_q = \sum_{p=1}^{N} a_p \omega^{(p-1)(q-1)}, \quad q = 1, \ldots, N, \ \omega \equiv e^{\frac{j2\pi}{N}} \tag{A1}$$

and the corresponding eigenvectors by

$$\{\mathrm{g}\}_q \equiv \frac{1}{\sqrt{N}} \left[1, \omega^{(q-1)}, \omega^{2(q-1)}, \ldots, \omega^{(N-1)(q-1)}\right]^T. \tag{A2}$$

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